Linear Algebra II

07/04/2022, Thursday, 16:00 – 18:00

1 $(7+7+6=20 \text{ pts})$ **Inner product spaces**

Let $\mathcal{V}$ be a finite-dimensional vector space with inner product $\langle\cdot,\cdot\rangle : \mathcal{V}\times\mathcal{V}\to\mathbb{R}$. Let $\{u_1, u_2, \ldots, u_n\}$ be an orthonormal basis of $\mathcal{V}$.

(a) Show that any vector $v\in\mathcal{V}$ can be expressed as

$$v = \langle v, u_1\rangle\, u_1 + \langle v, u_2\rangle\, u_2 + \cdots + \langle v, u_n\rangle\, u_n.$$

(b) Let $\varphi : \mathcal{V}\to\mathbb{R}$ be a linear map. That is, $\varphi(c_1w_1 + c_2w_2) = c_1\varphi(w_1) + c_2\varphi(w_2)$ for all $c_1, c_2\in\mathbb{R}$ and $w_1, w_2\in\mathcal{V}$. Show that there exists a vector $u\in\mathcal{V}$ such that

$$\varphi(v) = \langle u, v\rangle$$

for all $v\in\mathcal{V}$.

(c) Show that the vector u in (b) is *unique*. In other words, prove that if

$$\varphi(v) = \langle u_1, v\rangle \text{ and } \varphi(v) = \langle u_2, v\rangle \text{ for all } v\in\mathcal{V}$$

then $u_1 = u_2$.

2 $(6+12+6=24 \text{ pts})$ **Singular value decomposition**

Consider the matrix A given by

$$\begin{bmatrix} 1 & 0 \\ 2 & -2 \\ 0 & 1 \end{bmatrix}.$$

(a) Show that the singular values of A are $\sigma_1 = 3$ and $\sigma_2 = 1$.

(b) Find a singular value decomposition of A.

(c) Compute the best rank 1 approximation of A.

3 $(4+5+5+5+5=24$ pts) **Positive semi-definite matrices**

Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix. The notation $A \geqslant 0$ means: A is positive semi-definite.

(a) Prove that if $A \geqslant 0$ then all its eigenvalues λ satisfy $\lambda \geqslant 0$.

(b) Also prove the converse: if all eigenvalues λ of A satisfy $\lambda \geqslant 0$ then $A \geqslant 0$.

(c) For $r = 0, 1, \ldots n$, let A_r be the leading principal submatrices of A. Prove that if $A \geqslant 0$ then $A_r \geqslant 0$ for all r.

(d) Prove that if $A \geqslant 0$ then $\det(A_r) \geqslant 0$ for all r.

(e) Does the converse implication of statement (d) also holds? If not, give a counterexample.

4 $(4+5+3+3+4+3=22$ pts) **Jordan normal form**

Suppose $A \in \mathbb{C}^{n\times n}$ and let $p_A(z)$ be its characteristic polynomial.

(a) Let $q(z)$ be a polynomial that annihilates A. Prove that every eigenvalue λ of A is a root of $q(z)$, i.e. $q(\lambda) = 0$.

(b) Let $p_{\min}(z)$ be the minimal polynomial of A. Prove that $p_{\min}(z)$ is a divisor of $p_A(z)$, i.e. there exists a polynomial $d(z)$ such that $p_A(z) = d(z)p_{\min}(z)$.

(c) Prove that every root of $p_{\min}(z)$ is an eigenvalue of A.

Now let $n = 4$ and define

$$A := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix}.$$

(d) Determine the eigenvalues of A and their geometric multiplicities.

(e) Determine the Jordan normal form of A.

(f) Determine the minimal polynomial of A.

10 pts free